Solution of Systems with Toeplitz Matrices Generated by Rational Functions

William F. Trench
Department of Mathematics and Computer Science
Drexel University
Philadelphia, Pennsylvania 19104

Submitted by Hans Schneider

ABSTRACT

We consider Toeplitz matrices $T_n = (t_{i-j})_{i,j=0}^n$, where $\sum_{-\infty}^{\infty} t_j z^j$ is a formal Laurent series of a rational function $R(z)$. A criterion is given for T_n to be invertible, in terms of the nonvanishing of a determinant D_n involving the zeros of $R(z)$, and of order and form independent of n; i.e., n enters into D_n as a parameter, and not so as to complicate D_n as n increases. Explicit formulas involving similar determinants are given for the solution of the system $T_n X = Y$ in the case where T_n is invertible. Formulas are also given for T_n^{-1} in the case where T_{n-1} and T_n are both invertible. Suggestions concerning possible computational procedures based on the results are included.

1. INTRODUCTION

We consider the system

$$T_n X = Y, \tag{1}$$

where T_n is the Toeplitz matrix

$$T_n = (t_{i-j})_{i,j=0}^n \tag{2}$$

and $\{t_j\}$ is generated formally by a rational function. To be specific, let

$$A(z) = \sum_{\mu=0}^{r} a_\mu z^\mu, \qquad B(z) = \sum_{\nu=0}^{s} b_\nu z^\nu, \tag{3}$$

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and

$$C(z)=\sum_{l=-q}^{p} c_l z^l, \tag{4}$$

where $p,\ q,\ r,\ s\geqslant 0$ and $a_0a_rb_0b_sc_{-q}c_p\neq 0$, and it is assumed throughout that $A(z)$ and $z^sB(1/z)$ are relatively prime, so that there are unique polynomials $f(z)$ and $g(z)$ of degree less than s and r, respectively, such that

$$1=f(z)A(z)+g(z)z^sB(1/z). \tag{5}$$

Now suppose that R_1 and R_2 are the minima of the moduli of the zeros of $A(z)$ and $B(z)$, respectively, and expand

$$[A(z)]^{-1}=\sum_{\mu=0}^{\infty} P_\mu z^\mu, \qquad |z|<R_1, \tag{6}$$

and

$$\left[B\left(\frac{1}{z}\right)\right]^{-1}=\sum_{\nu=0}^{\infty} Q_\nu z^{-\nu}, \qquad |z|>\frac{1}{R_2}. \tag{7}$$

Now define the formal Laurent series

$$\Phi(z)=\sum_{j=-\infty}^{\infty}\phi_j z^j=f(z)\left(\sum_{\nu=0}^{\infty}Q_\nu z^{-\nu}\right)+z^sg(z)\left(\sum_{\mu=0}^{\infty}P_\mu z^\mu\right) \tag{8}$$

and

$$T(z)=\sum_{j=-\infty}^{\infty}t_jz^j=C(z)\Phi(z); \tag{9}$$

i.e., let

$$t_j=\sum_{l=-q}^{p}c_l\phi_{j-l}. \tag{10}$$

From (5), (6), (7), (8), and (9), $T(z)$ is a formal Laurent series for the rational function

$$R(z)=\frac{C(z)}{A(z)B(1/z)}. \tag{11}$$

If $R_1R_2>1$, then $T(z)$ converges to $R(z)$ for $1/R_2<|z|<R_1$; however, only the formal definition—and not convergence—is assumed below.

Hermitian Toeplitz matrices of this kind occur as covariance matrices of autoregressive moving average stationary time series; in fact, the approach used here has its origins in the author's consideration of an estimation problem for such time series [7, Sections 4, 5].

In the following we consider matrices (2) with

$$n \geqslant \max(0,\, r+s-p-q), \tag{12}$$

and let

$$k=p+q.$$

In Section 2 we reduce (1) to a related boundary problem for a difference equation, and we show that T_n is invertible if and only if a certain $k\times k$ determinant D_n is nonzero. The entries in D_n are polynomials in the zeros $z_1,\dots,z_k$ of the Laurent polynomial $C(z)$, and the form of D_n is independent of n; i.e., n enters into D_n as a parameter, and not in such a way as to make D_n more complicated with increasing n. Section 3 contains formulas for the solution of (1) in the case where T_n is invertible, again in terms of determinants of the same general type as D_n. We do not claim that these formulas constitute an efficient way to actually solve (1) numerically, but we do believe that they offer new insight into the properties of these matrices, and that they may be useful for theoretical purposes.

In Section 4 we present an algorithm for computing the solutions of (1). The algorithm requires that the zeros of $C(z)$ be known. In part of the computation n occurs as a parameter, so that the cost of this part is independent of n. The cost of the remaining computation is proportional to n. Since it does not involve recursion with respect to n, the algorithm can be used to solve (1) for a given n such that T_n is invertible, without requiring that T_k be invertible for any $k\neq n$. However, the question of stability of this algorithm remains open.

Section 5 presents a method for inverting T_n provided T_n and T_{n-1} are both invertible.

This paper extends and improves results obtained in [8] for the case where $A(z)=B(z)=1$, so that T_n is a banded Toeplitz matrix. Many papers have dealt with solving banded Toeplitz systems (see the bibliographies of [8] and its references); however, very little work has been devoted to solving Toeplitz systems associated with more general rational generating functions. To the author's knowledge, only Dickinson [2] has stated a specific algorithm for solving such systems. His algorithm yields the solution of (1) in $O(n)$ operations, provided that $T_0,\dots,T_n$ are all invertible. Under assumptions more restrictive than ours, Day [1] has given a formula for $\det T_n$ in terms of the zeros of the numerator and denominator of $R(z)$ [cf. (11)].

2. A RELATED BOUNDARY VALUE PROBLEM

We write

$$A(z)B(1/z)=\sum_{j=-s}^{r}\theta_j z^j \tag{13}$$

and define

$$a_\mu=0,\quad b_\nu=0,\quad \theta_j=0,\quad c_l=0$$

for values of μ, ν, j, and l outside the limits of the sums in (3), (4), and (13). We also define

$$\sum_i^j=0 \qquad \text{if} \quad i>j.$$

With these conventions, for example,

$$\theta_j=\sum_{\nu=0}^{s}a_{j+\nu}b_\nu=\sum_{\mu=0}^{r}a_\mu b_{\mu-j},\qquad -\infty<j<\infty. \tag{14}$$

If $C(z)=1$, then (10) implies that

$$T_m=\Phi_m=\left(\phi_{i-j}\right)_{i,j=0}^{m},$$

for which an explicit inversion formula is already known, as follows. (See [4] and [5].)

LEMMA 1 (Greville and Trench). *If* $a_0 b_0 \neq 0$, $A(z)$ *and* $z^s B(1/z)$ *are relatively prime, and* $m \geqslant r+s$, *then*

$$\Phi_m^{-1} = \left(\psi_{ijm}\right)_{i,j=0}^{m},$$

where

$$\psi_{ijm} = \theta_{i-j} - \sum_{\nu=j+1}^{s} a_{i-j+\nu} b_\nu - \sum_{\mu=m+1-j}^{r} b_{j-i+\mu} a_\mu .$$

This lemma and elementary (but tedious) manipulations based on (14) yield the following lemma.

LEMMA 2. *If the assumptions of Lemma* 1 *hold and*

$$\eta_i = \sum_{j=0}^{m} \phi_{i-j} \xi_j, \qquad 0 \leqslant i \leqslant m, \tag{15}$$

then

$$\xi_i = \sum_{\nu=0}^{i} a_\nu \sum_{j=0}^{s} b_j \eta_{i-\nu+j}, \qquad 0 \leqslant i \leqslant m-s, \tag{16}$$

and

$$\xi_i = \sum_{\mu=0}^{m-i} b_\mu \sum_{j=0}^{r} a_j \eta_{i+\mu-j}, \qquad r \leqslant i \leqslant m. \tag{17}$$

Moreover, the formulas in (16) *and* (17) *both reduce to*

$$\xi_i = \sum_{j=-s}^{r} \theta_j \eta_{i-j} \qquad \text{for} \quad r \leqslant i \leqslant m-s. \tag{18}$$

In the following it is to be understood that any condition imposed for $\mu \leqslant i \leqslant \nu$ is vacuous if $\mu > \nu$.

LEMMA 3. *Suppose* (11) *holds and the vectors*

$$U=[u_{-p},\dots,u_{n+q}] \quad \text{and} \quad \hat{X}=[x_{-p},\dots,x_{n+q}]$$

are related by

$$u_i=\sum_{j=-p}^{n+q}\phi_{i-j}x_j, \qquad -p\leqslant i\leqslant n+q. \tag{19}$$

Then

$$x_i=0 \qquad \text{for} \quad -p\leqslant i\leqslant -1 \tag{20}$$

if and only if

$$\sum_{l=0}^{s}b_l u_{l-i}=0, \qquad 1\leqslant i\leqslant p, \tag{21}$$

and

$$x_i=0 \qquad \text{for} \quad n+1\leqslant i\leqslant n+q \tag{22}$$

if and only if

$$\sum_{l=0}^{r}a_l u_{i-l}=0, \qquad n+1\leqslant i\leqslant n+q. \tag{23}$$

Proof. We apply Lemma 2 with $m=n+p+q$, $\xi_i=x_{-p+i}$, and $\eta_i=u_{-p+i}$. Then (19) is equivalent to (15), so (16) and (17) imply that

$$x_{-p+i}=\sum_{\nu=0}^{i}a_\nu\sum_{j=0}^{s}b_j u_{i-\nu-p+j}, \qquad 0\leqslant i\leqslant n+p+q-s, \tag{24}$$

and

$$x_{-p+i}=\sum_{\mu=0}^{n+p+q-i}b_\mu\sum_{j=0}^{r}a_j u_{i+\mu-p-j}, \qquad r\leqslant i\leqslant n+p+q. \tag{25}$$

Since $a_0 b_0 \neq 0$, (24) implies the equivalence of (20) and (21), while (25) implies the equivalence of (22) and (23). ■

Note that (18) implies that

$$x_{-p+i} = \sum_{j=-s}^{r} \theta_j u_{i-p-j}, \qquad r \leqslant i \leqslant n+p+q-s. \tag{26}$$

THEOREM 1. *If* (12) *holds, then the system*

$$\sum_{j=0}^{n} t_{i-j} x_j = y_i, \qquad 0 \leqslant i \leqslant n, \tag{27}$$

has a solution $x_0, \dots, x_n$ *if and only if the system*

$$\sum_{l=-q}^{p} c_l u_{i-l} = y_i, \qquad 0 \leqslant i \leqslant n, \tag{28a}$$

$$\sum_{l=0}^{r} a_l u_{i-l} = 0, \qquad n+1 \leqslant i \leqslant n+q, \tag{28b}$$

$$\sum_{l=0}^{s} b_l u_{l-i} = 0, \qquad 1 \leqslant i \leqslant p, \tag{28c}$$

has a solution $u_{-p}, \dots, u_{n+q}$. *If this is so, then the solution of* (27) *is given by*

$$x_i = \sum_{\nu=0}^{i} a_\nu \sum_{j=0}^{s} b_j u_{i-\nu+j}, \qquad 0 \leqslant i \leqslant r-p-1, \tag{29}$$

$$x_i = \sum_{j=-s}^{r} \theta_j u_{i-j}, \qquad \max(0, r-p) \leqslant i \leqslant \min(n, n+q-s), \tag{30}$$

$$x_i = \sum_{\mu=0}^{n-i} b_\mu \sum_{j=0}^{r} a_j u_{i+\mu-j}, \qquad n+q-s+1 \leqslant i \leqslant n. \tag{31}$$

Proof. Because of (10), (27) is equivalent to (28a), with

$$u_i = \sum_{j=0}^{n} \phi_{i-j} x_j, \qquad -p \leqslant i \leqslant n+q. \tag{32}$$

Now suppose (27) has a solution $x_0,\dots,x_n$, and define $u_{-p},\dots,u_{n+q}$ by (32). Then (28a) holds, and (32) is of the form (19), subject to (20) and (22). Hence, Lemma 3 implies (28b) and (28c). Conversely, if $u_{-p},\dots,u_{n+q}$ satisfy (28), then define $x_{-p},\dots,x_{n+q}$ as the solution of (19). Then (28b) and (28c) imply (20) and (22), so (19) reduces to (32), and the opening sentence of the proof implies (27).

To obtain (29), (30), and (31), replace i by $p+i$ in (24), (25), and (26), and recall (28b) and (28c). ∎

Theorem 1 reduces the problem of solving (1) to solving the boundary value problem (28) for $u_{-p},\dots,u_{n+q}$ and then computing $x_0,\dots,x_n$ from (29), (30), and (31). We will now use this to derive a criterion for invertibility of T_n. In Section 3 we obtain the explicit solution of the boundary value problem.

The following definition applies throughout the remainder of the paper.

Definition 1. Let $z_1,\dots,z_L$ be the distinct zeros of the Laurent polynomial $C(z)$ [cf. (4)], with multiplicities $\mu_1,\dots,\mu_L$; thus, $L\leqslant k$ and

$$\mu_1+\cdots+\mu_L=k\ (=p+q).$$

If $Q(z)$ is an arbitrary Laurent polynomial, define the k-dimensional row vector generated by $Q(z)$ as follows: its first μ_1 entries are $Q^{(l)}(z_1)$, $0\leqslant l\leqslant \mu_1-1$; its next μ_2 entries are $Q^{(l)}(z_2)$, $0\leqslant l\leqslant\mu_2-1$; and so forth. If $Q_1(z),\dots,Q_k(z)$ are k Laurent polynomials, let

$$\Omega[Q_1(z),\dots,Q_k(z)] \tag{33}$$

be the matrix whose μth row $(1\leqslant\mu\leqslant k)$ is generated by $Q_\mu(z)$, and let

$$D[Q_1(z),\dots,Q_k(z)] \tag{34}$$

be the determinant of this matrix. Thus, if $C(z)$ has distinct zeros $z_1,\dots,z_k$, then

$$\Omega[Q_1(z),\dots,Q_k(z)]=\left(Q_\mu(z_\nu)\right)_{\mu,\nu=1}^{k}.$$

The rather cumbersome notation in (33) and (34) will be convenient below, where it will often be necessary to make the choice of $Q_1(z),\dots,Q_k(z)$ explicit. It should be noted that although the notation suggests that Ω and D

are functions of z, they are not. Also, the fact that the columns of Ω are permuted if the zeros of $C(z)$ are renumbered will not produce ambiguities in our results.

THEOREM 2. *Let Ω_n and D_n denote the $k \times k$ matrix and determinant, respectively, that result from Definition* 1 *when*

$$Q_i(z) = \begin{cases} z^{i-1}A(z), & 1 \leqslant i \leqslant q, \\ z^{n+i}B(1/z), & q+1 \leqslant i \leqslant k. \end{cases} \tag{35}$$

Then T_n is singular for some n which satisfies (12) *if and only if $D_n = 0$.*

Proof. The matrix T_n is singular if and only if $T_n X = 0$ for some nonzero X. This and Theorem 1 imply that T_n is singular if and only if there are numbers $u_{-p}, \dots, u_{n+q}$, not all zero, which satisfy (28b), (28c), and the homogeneous system

$$\sum_{l=-q}^{p} c_l u_{i-l} = 0, \qquad 0 \leqslant i \leqslant n. \tag{36}$$

But we can obtain the general solution of (36) easily, as follows. The hypotheses of Definition 1 imply that for every integer i, z_j is a zero of $z^{n+q-i}C(z)$, with multiplicity μ_j. Hence, if $(m)^{(0)} = 1$ and $(m)^{(\nu)} = m(m-1)\cdots(m-\nu+1)$ $(\nu \geqslant 1)$ when m is an integer, then

$$\sum_{l=-q}^{p} c_l (n+q-i+l)^{(\nu)} z_j^{n+q-i+l-\nu} = 0,$$

$$0 \leqslant \nu \leqslant \mu_j - 1, \quad 1 \leqslant j \leqslant L, \quad -\infty < i < \infty,$$

since the sum on the left equals

$$\left[z^{n+q-i} C(z) \right]^{(\nu)} \Big|_{z=z_j}.$$

This means that the k sequences

$$\left\{ (n+q-i)^{(\nu)} z_j^{n+q-i-\nu} \Big| -p \leqslant i \leqslant n+q \right\}, \qquad 0 \leqslant \nu \leqslant \mu_j - 1, \quad 1 \leqslant j \leqslant L,$$

all satisfy (36). Since they are linearly independent, the general solution of (36) is of the form

$$u_i = \sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \alpha_{\nu j}(n+q-i)^{(\nu)} z_j^{n+q-i-\nu}, \qquad -p \leqslant i \leqslant n+q. \quad (37)$$

Substituting (37) into (28b) and summing the result first on l yields

$$\sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \alpha_{\nu j} \left[z^{n+q-i} A(z) \right]^{(\nu)} \Big|_{z=z_j} = 0, \qquad n+1 \leqslant i \leqslant n+q. \quad (38)$$

Substituting (37) into (28c) and summing the result first on l yields

$$\sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \alpha_{\nu j} \left[z^{n+q+i} B(1/z) \right]^{(\nu)} \Big|_{z=z_j} = 0, \qquad 1 \leqslant i \leqslant p. \quad (39)$$

Thus, $T_n X = 0$ for some nonzero X if and only if the system consisting of the k equations (38) and (39) has a nontrivial solution

$$\left[\alpha_{01}, \ldots, \alpha_{\mu_1 - 1, 1}, \ldots, \alpha_{L1}, \ldots, \alpha_{L, \mu_L - 1} \right].$$

However, after a suitable reordering of these equations, we see that Ω_n is the matrix of the system (38) and (39). This implies the conclusion. ∎

Henceforth, we assume that the determinant

$$D_n = D\left[A(z), \ldots, A(z) z^{q-1},\ z^{n+q+1} B(1/z), \ldots, z^{n+k} B(1/z) \right]$$

is nonzero, so that T_n is invertible. It should be observed that if $p = 0$, then D_n reduces to

$$D_n = D\left[A(z), \ldots, z^{q-1} A(z) \right],$$

while if $q = 0$, then

$$D_n = D\left[z^{n+1} B(1/z), \ldots, z^{n+p} B(1/z) \right].$$

(The case where $p = q = 0$ is of no interest, because of Lemmas 1 and 2.)

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

The following lemma indicates the way in which we approach the boundary value problem. We omit the proof, which is routine.

LEMMA 4. *Suppose $D_n \neq 0$, and let*

$$V = [v_{-p}, \dots, v_{n+q}]$$

be any solution of the system

$$\sum_{l=-q}^{p} c_l v_{i-l} = y_i, \qquad 0 \leqslant i \leqslant n. \tag{40}$$

Define $A_1, \dots, A_q$ and $B_1, \dots, B_p$ by

$$A_i = -\sum_{l=0}^{r} a_l v_{n+i-l}, \qquad 1 \leqslant i \leqslant q, \tag{41}$$

and

$$B_i = -\sum_{l=0}^{s} b_l v_{l-i}, \qquad 1 \leqslant i \leqslant p. \tag{42}$$

Now let

$$W = [w_{-q}, \dots, w_{n+p}] \tag{43}$$

be the (unique) solution of the homogeneous boundary value problem

$$\sum_{l=-q}^{p} c_l w_{i-l} = 0, \qquad 0 \leqslant i \leqslant n, \tag{44a}$$

$$\sum_{l=0}^{r} a_l w_{i-l} = A_{i-n}, \qquad n+1 \leqslant i \leqslant n+q, \tag{44b}$$

$$\sum_{l=0}^{s} b_l w_{l-i} = B_i, \qquad 1 \leqslant i \leqslant p. \tag{44c}$$

Then the (unique) solution

$$U=[u_{-q},\dots,u_{n+p}]$$

of (28) *is* $U=V+W$.

To solve (40) we need only find a formal reciprocal of $C(z)$, i.e., a formal Laurent series

$$\Gamma(z)=\sum_{j=-\infty}^{\infty}\gamma_j z^j$$

such that

$$\Gamma(z)C(z)=1. \tag{45}$$

Then the sequence

$$v_i=\sum_{j=0}^{n}y_j\gamma_{i-j}, \qquad -p\leqslant i\leqslant n+q, \tag{46}$$

satisfies (40). To see this, let

$$Y(z)=\sum_{j=0}^{n}y_j z^j$$

and

$$V(z)=\Gamma(z)Y(z)=\sum_{i=-\infty}^{\infty}v_i z^i.$$

Then (46) holds, and (45) implies that $V(z)C(z)=Y(z)$, which in turn implies (40).

There are in general many formal reciprocals for $C(z)$. One can be obtained by expanding $[z^qC(z)]^{-1}$ in nonnegative powers of z:

$$[z^qC(z)]^{-1}=\sum_{j=0}^{\infty}\alpha_j z^j, \qquad |z|<R=\min_{1\leqslant\nu\leqslant L}|z_\nu|.$$

Then the series

$$\Gamma(z)=\sum_{j=q}^{\infty}\alpha_{j-q}z^j \tag{47}$$

satisfies (45), and (46) becomes

$$v_i = \begin{cases} 0, & -p \leqslant i \leqslant q-1, \\ \sum_{j=0}^{i-q} \alpha_{i-j-q} y_j, & q \leqslant i \leqslant n+q. \end{cases} \tag{48}$$

Another formal reciprocal for $C(z)$ can be obtained by expanding $[z^{-p}C(z)]^{-1}$ in nonpositive powers of z:

$$[z^{-p}C(z)]^{-1} = \sum_{j=0}^{\infty} \beta_j z^{-j}, \qquad |z| > \frac{1}{R}.$$

Then the series

$$\Gamma(z) = \sum_{j=-\infty}^{-p} \beta_{-j-p} z^j \tag{49}$$

satisfies (45), and (46) becomes

$$v_i = \begin{cases} \sum_{j=i+p}^{n} \beta_{j-i-p} y_j, & -p \leqslant i \leqslant n-p, \\ 0, & n-p+1 \leqslant i \leqslant n+q. \end{cases} \tag{50}$$

Obviously, the sequences $\{\alpha_r\}$ and $\{\beta_r\}$ can be computed recursively. They can also be represented explicitly in terms of the zeros of $C(z)$; in the notation of Definition 1,

$$\alpha_j = \frac{D[z^{-j}, 1, z, \dots, z^{k-1}]}{c_{-q} D[1, z, \dots, z^{k-1}]}, \qquad j \geqslant 0,$$

and

$$\beta_j = \frac{D[1, z, \dots, z^{k-2}, z^{j+k-1}]}{c_p D[1, z, \dots, z^{k-1}]}, \qquad j \geqslant 0.$$

(See [8, Lemma 2].)

It should be borne in mind that (48) and (50) are different solutions of (40). Because of the "triangular" form of (48) and (50), it would appear that (47) and (49) are the most efficient choices for $\Gamma(z)$ from a computational point of view. However, these choices do not exhaust the possibilities. For example, suppose we rewrite $C(z)$ as

$$C(z) = z^{p-r'}A_1(z)B_1(1/z),$$

where $\deg A_1(z) = r'$, $\deg B_1(z) = s'$ $(= k - r')$, and $A_1(z)$ and $B_1(1/z)$ have no zeros in common. Now obtain the formal reciprocal $\Psi(z)$ for $A_1(z)B_1(1/z)$ in the same way that we obtained $\Phi(z)$ for $A(z)B(1/z)$ in (5), (6), (7), and (8); then the series $\Gamma(z) = z^{r'-p}\Psi(z)$ satisfies (45).

There are still other ways to choose $\Gamma(z)$, since if $\Gamma_1(z), \ldots, \Gamma_m(z)$ are formal reciprocals of $C(z)$ and $h_1, \ldots, h_m$ are constants with sum equal to one, then the formal series

$$\Gamma(z) = h_1\Gamma_1(z) + \cdots + h_m\Gamma_m(z)$$

satisfies (45).

We now present two ways for finding the vector W in (43). The first is applicable when the main interest is in numerical computation. The second is of theoretical interest, since it provides explicit formulas for the solution of (44).

As we saw in the proof of theorem 2, the general solution of (44a) is of the form

$$w_i = \sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \beta_{\nu j}(n+q-i)^{(\nu)} z_j^{n+q-i-\nu}, \qquad -p \leqslant i \leqslant n+q. \tag{51}$$

Therefore, solving (44) reduces to choosing the k coefficients $\{\beta_{\nu j}\}$ so as to satisfy the boundary conditions (44b) and (44c). Manipulations like those which yielded (38) and (39) lead to the equations

$$\sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \beta_{\nu j} \left[z^{n+q-i}A(z)\right]^{(\nu)}\Big|_{z=z_j} = A_i, \qquad n+1 \leqslant i \leqslant n+q, \tag{52a}$$

$$\sum_{j=1}^{L} \sum_{\nu=0}^{\mu_j - 1} \beta_{\nu j} \left[z^{n+q+i}B(1/z)\right]^{(\nu)}\Big|_{z=z_j} = B_i, \qquad 1 \leqslant i \leqslant p. \tag{52b}$$

Except for a permutation of its first q rows, the determinant of this system is D_n, which is assumed to be nonzero; therefore, the solution of the boundary value problem (44) reduces to solving a $k \times k$ system.

We will now derive explicit formulas for the solution of (44). To this end, we define $e_{\mu n}(j)$ $(1 \leqslant \mu \leqslant k)$ to be the $k \times k$ determinant obtained by replacing the generator of the μ-th row of D_n with z^{n+q-j}; i.e.,

$$e_{\mu n}(j) = D[Q_1(z), \ldots, Q_k(z)],$$

with $Q_i(z)$ as in (35) if $i \neq \mu$, and $Q_\mu(z) = z^{n+q-j}$.

LEMMA 5. *Suppose $1 \leqslant \mu \leqslant k$. Then the sequence $\{e_{\mu n}(j)\}_{j=-\infty}^{\infty}$ satisfies the following equations*:

$$\sum_{l=-q}^{p} c_l e_{\mu n}(j-l) = 0, \qquad -\infty < j < \infty, \tag{53}$$

$$\sum_{l=0}^{r} a_l e_{\mu n}(j-l) = \delta_{\mu, n+q+1-j} D_n, \qquad n+1 \leqslant j \leqslant n+q, \tag{54}$$

$$\sum_{l=0}^{s} b_l e_{\mu n}(l-j) = \delta_{\mu, q+j} D_n, \qquad 1 \leqslant j \leqslant p. \tag{55}$$

Proof. The left sides of the last three equations can be viewed as determinants of the form (34) with $Q_i(z)$ as in (35) [i.e., $Q_i(z)$ is the generator of the ith row of D_n] if $i \neq \mu$, and

$$Q_\mu(z) = z^{n+q-j}C(z) \qquad \text{for (53)}, \tag{56}$$

$$Q_\mu(z) = z^{n+q-j}A(z) \qquad \text{for (54)}, \tag{57}$$

$$Q_\mu(z) = z^{n+q+j}B(1/z) \qquad \text{for (55)}. \tag{58}$$

Since $z_1, \ldots, z_L$ are zeros of $z^{n+q-j}C(z)$ with multiplicities $\mu_1, \ldots, \mu_L$ for every j, (56) implies that the μth row of the determinant on the left of (53) consists entirely of zeros. This implies (53). If $n+1 \leqslant j \leqslant n+q$, then the polynomial in (57) is the same as the generator of row $n+q+1-j$ of D_n; therefore, the determinant on the left of (54) has two identical rows unless $\mu = n+q+1-j$, in which case it equals D_n. This implies (54). If $1 \leqslant j \leqslant p$, then the polynomial in (58) is the generator of row $q + j$ of D_n; therefore, the determinant on the left of (55) has two identical rows unless $\mu = q + j$, in which case it equals D_n. This implies (55). ∎

It is easily verified that Lemma 5 implies the following theorem.

THEOREM 3. *The solution to the boundary value problem* (44) *is given by*

$$w_i = \frac{1}{D_n}\left[\sum_{\mu=1}^{q} A_{q+1-\mu} e_{\mu n}(i) + \sum_{\mu=q+1}^{k} B_{\mu-q} e_{\mu n}(i)\right], \qquad -q \leqslant i \leqslant n-p. \tag{59}$$

4. AN ALGORITHM FOR SOLVING $T_n X = Y$

In this section we assume that the zeros of $C(z)$ are known, and that $D_n \neq 0$. We can then solve (1) by means of the following steps.

(A) Obtain $v_{-p}, \ldots, v_{n+q}$ by defining

$$v_i = 0, \qquad -p \leqslant i \leqslant q-1, \tag{60}$$

and then computing recursively from (40),

$$v_i = \frac{1}{c_{-q}}\left[y_{i-q} - \sum_{l=-q+1}^{p} c_l v_{i-q-l}\right], \qquad q \leqslant i \leqslant n+q. \tag{61}$$

or by defining

$$v_i = 0, \qquad n-p+1 \leqslant i \leqslant n+q, \tag{62}$$

and computing recursively from (40),

$$v_{n-i} = \frac{1}{c_p}\left[y_{n+p-i} - \sum_{l=-q}^{p-1} c_l v_{n+p-i-l}\right], \qquad p \leqslant i \leqslant n+p. \tag{63}$$

[Solving (60) and (61) explicitly yields (48), while solving (62) and (63) explicitly yields (50).]

(B) Compute $A_1,\dots,A_q$ and $B_1,\dots,B_p$ from (41) and (42). Then solve the $k\times k$ system (52) for the coefficients $\{\beta_{\nu j}\}$ in (51).

(C) Obtain $w_{-p},\dots,w_{n+q}$ either by computing $w_{-p},\dots,w_{q-1}$ explicitly from (51) and then computing recursively from (44a),

$$w_i = -\frac{1}{c_{-q}}\sum_{l=-q+1}^{p} c_l w_{i-q-l}, \qquad q\leqslant i\leqslant n+q,$$

or by computing $w_{n-p+1},\dots,w_{n+q}$ explicitly from (51) and then computing recursively from (44a),

$$w_i = -\frac{1}{c_p}\sum_{l=-q}^{p-1} c_l w_{n+p-i-l}, \qquad p\leqslant i\leqslant n+p.$$

(D) Compute

$$u_i = v_i + w_i, \qquad -p\leqslant i\leqslant n+q. \tag{64}$$

(E) Compute $x_0,\dots,x_n$ from (29), (30), and (31).

5. FORMULAS FOR T_n^{-1}

In this section we give formulas for obtaining the inverse

$$T_n^{-1} = \left(h_{ijn}\right)_{i,j=0}^{n}, \tag{65}$$

of an invertible Toeplitz matrix (2) of the type studied here, in the case where

$$h_{00n}\neq 0, \tag{66}$$

which is equivalent to assuming that T_{n-1} is also invertible. We need the following lemma, which is clearly implicit in the last four equations of [6].

LEMMA 6. *Let T_n be a Toeplitz matrix with inverse* (65) *which satisfies* (66). *Then*

$$h_{ijn} = h_{i-1,j-1,n} + \left(h_{00n}\right)^{-1}\left[h_{i0n}h_{0jn} - h_{n-j+1,0,n}h_{0,n-i+1,n}\right], \qquad 1\leqslant i,j\leqslant n. \tag{67}$$

Since T_n^{-1} is persymmetric (i.e., symmetric about its secondary diagonal), it is only necessary to use (67) for $i+j\leqslant n$, and then take

$$h_{ijn}=h_{n-j,n-i,n},\qquad 1\leqslant i,j\leqslant n,\quad i+j>n.$$

The formula (67) was rediscovered and presented in a useful matrix form by Gohberg and Semencul [3]. It provides an efficient means for obtaining T_n^{-1} once its first (zeroth) row and column are known. Therefore, coupling the next two theorems with Lemma 6 yields, in principle, an efficient way to obtain T_n^{-1} when T_n is rationally generated and $D_nD_{n-1}\neq 0$.

THEOREM 4. *Suppose that* (12) *holds*, $D_n\neq 0$, *and*

$$n\geqslant r-p. \tag{68}$$

Then the elements of the zeroth column of T_n^{-1} *are given by*

$$h_{i0n}=-\frac{b_0}{c_pD_n}D\left[\ldots,\ z^{n+q-i}B\left(\frac{1}{z}\right)\sum_{\nu=0}^{i}a_\nu z^\nu\right],\qquad 0\leqslant i\leqslant r-p-1, \tag{69}$$

$$h_{i0n}=\frac{\theta_r\delta_{i,r-p}}{c_p}-\frac{b_0}{c_pD_n}D\left[\ldots,\ z^{n+q-i}A(z)B\left(\frac{1}{z}\right)\right],$$

$$\max(0,r-p)\leqslant i\leqslant\min(n,n+q-s), \tag{70}$$

$$h_{i0n}=-\frac{b_0}{c_pD_n}D\left[\ldots,\ z^{n+q-i}A(z)\sum_{\mu=0}^{n-i}b_\mu z^{-\mu}\right],\qquad n+q-s+1\leqslant i\leqslant n, \tag{71}$$

where "..." denotes

$$A(z),\ldots,z^{q-1}A(z),z^{n+q+1}B(1/z),\ldots,z^{n+k-1}B(1/z),$$

with appropriate modifications if $p=1$ *or* $q=0$.

Proof. The zeroth column of T_n^{-1} is the solution of (1) with

$$y_i=\delta_{i0},\qquad 0\leqslant i\leqslant n.$$

Here we use (63) to define V, which yields

$$v_i = \frac{\delta_{-p,i}}{c_p}, \qquad -p \leqslant i \leqslant n+q. \tag{72}$$

Now (41) [with (68)], (42), and (72) imply that

$$A_i = 0, \quad 1 \leqslant i \leqslant q; \qquad B_i = -\frac{b_0 \delta_{p,i}}{c_p}, \quad 1 \leqslant i \leqslant p.$$

Therefore,

$$w_i = -\frac{b_0}{c_p D_n} e_{kn}(i), \qquad -p \leqslant i \leqslant n+q,$$

from (59). This, (64), and (72) imply that

$$u_i = \frac{\delta_{-p,i}}{c_p} - \frac{b_0}{c_p D_n} e_{kn}(i), \qquad -p \leqslant i \leqslant n+q. \tag{73}$$

Because of the definition of $e_{kn}(i)$, substituting (73) into (29), (30), and (31) (where $x_i = h_{i0_n}$) and invoking elementary properties of determinants of the kind used in the proof of Lemma 5 yields (69), (70), and (71).

REMARK 1. The determinants $D[\cdots]$ in (69), (70), and (71) have the same entries in their first $k-1$ rows as does D_n, while their kth rows are generated (in the sense of Definition 1) by the Laurent polynomials exhibited in the respective brackets. Moreover, the quantities

$$\lambda_{in} = D\left[\cdots, z^{n+q-1} A(z) B(1/z)\right] \tag{74}$$

appearing in (70) satisfy the difference equation

$$\sum_{l=-q}^{p} c_l \lambda_{i-l,n} = 0, \tag{75}$$

and therefore it is only necessary to calculate λ_{in} explicitly from the formula (74) for k successive values of i, since the rest can then be computed recursively from (75). (However, this method could introduce a stability problem.)

THEOREM 5. *Suppose that* (12) *holds*, $D_n \neq 0$, *and* $n \geqslant s - q$. *Then the elements of the zeroth row of* T_n^{-1} *are given by*

$$h_{0,n-i,n} = -\frac{a_0}{c_{-q}D_n} D\left[z^{n+q-i} B\left(\frac{1}{z}\right) \sum_{\nu=0}^{i} a_\nu z^\nu, \dots \right],$$

$$0 \leqslant i \leqslant r - p - 1, \quad (76)$$

$$h_{0,n-i,n} = \frac{\theta_{-s}\delta_{i,n+q-s}}{c_{-q}} - \frac{a_0}{c_{-q}D_n} D\left[z^{n+q-i} A(z) B\left(\frac{1}{z}\right), \dots \right],$$

$$\max(0, r-p) \leqslant i \leqslant \min(n, n+q-s), \quad (77)$$

$$h_{i0n} = -\frac{a_0}{c_{-q}D_n} D\left[z^{n+q-i} A(z) \sum_{\mu=0}^{n-i} b_\mu z^{-\mu}, \dots \right],$$

$$n + q - s + 1 \leqslant i \leqslant n, \quad (78)$$

where "..." *denotes*

$$zA(z), \dots, z^{q-1}A(z),\ z^{n+q+1}B(1/z), \dots, z^{n+k}B(1/z),$$

with appropriate modifications if $p = 0$ *or* $q = 1$.

Proof. Because of the persymmetry of T_n^{-1},

$$h_{0,n-i,n} = h_{nin}, \qquad 0 \leqslant i \leqslant n.$$

Therefore, the vector

$$X = \operatorname{col}[h_{0nn}, h_{0,n-1,n}, \dots, h_{00n}]$$

is the last column of T_n^{-1}, i.e., the solution of (1) with

$$y_i = \delta_{in}, \qquad 0 \leqslant i \leqslant n.$$

Now we use (61) to define v_i, which yields

$$v_i = \frac{\delta_{n+q,i}}{c_{-q}}, \qquad -p \leqslant i \leqslant n+q.$$

The remainder of the proof is similar to that of Theorem 4. ■

Remark 1 also applies to the determinants $D[\cdots]$ in (76), (77), and (78), with obvious modifications; i.e., the *last* $k-1$ rows of $D[\cdots]$ are the same as those of D_n, while the *first* is generated by the Laurent polynomial exhibited in the brackets.

REFERENCES

1 K. M. Day, Toeplitz matrices generated by the Laurent series expansion of an arbitrary rational function, *Trans. Amer. Math. Soc.* 206:224–245 (1975).
2 B. W. Dickinson, Solution of linear equations with rational Toeplitz matrices, *Math. Comp.* 34:227–233 (1980).
3 I. C. Gohberg and A. A. Semencul, On the inversion of finite Toeplitz matrices and their continuous analogs (in Russian), *Mat. Issled.* 2:201–233 (1972).
4 T. N. E. Greville, On a problem concerning band matrices with Toeplitz inverses, in *Proceedings of the 8th Manitoba Conference on Numerical Mathematics and Computation*, Utilitas Math., 1978, pp. 275–283.
5 T. N. E. Greville and W. F. Trench, Band matrices with Toeplitz inverses, *Linear Algebra Appl.* 27:199–209 (1979).
6 W. F. Trench, An algorithm for the inversion of finite Toeplitz matrices, *SIAM J. Appl. Math.* 12:515–522 (1964).
7 W. F. Trench, Weighting coefficients for the prediction of stationary time series from the finite past, *SIAM J. Appl. Math.* 15:1502–1510 (1967).
8 W. F. Trench, Explicit inversion formulas for Toeplitz band matrices, *SIAM J. Algebraic Discrete Methods*, to appear.

Received 22 September 1984; revised 18 December 1984